

RESTRICTED PERMUTATIONS, CONTINUED FRACTIONS, AND CHEBYSHEV POLYNOMIALS

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ABSTRACT. Let $f_n^r(k)$ be the number of 132-avoiding permutations on n letters that contain exactly r occurrences of $12 \dots k$, and let $F_r(x; k)$ and $F(x, y; k)$ be the generating functions defined by $F_r(x; k) = \sum_{n \geq 0} f_n^r(k) x^n$ and $F(x, y; k) = \sum_{r \geq 0} F_r(x; k) y^r$. We find an explicit expression for $F(x, y; k)$ in the form of a continued fraction. This allows us to express $F_r(x; k)$ for $1 \leq r \leq k$ via Chebyshev polynomials of the second kind.

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1. INTRODUCTION

Let $[p] = \{1, \dots, p\}$ denote a totally ordered alphabet on p letters, and let $\alpha = (\alpha_1, \dots, \alpha_m) \in [p_1]^m$, $\beta = (\beta_1, \dots, \beta_m) \in [p_2]^m$. We say that α is *order-isomorphic* to β if for all $1 \leq i < j \leq m$ one has $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$. For two permutations $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$, an *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $(\pi_{i_1}, \dots, \pi_{i_k})$ is order-isomorphic to τ ; in such a context τ is usually called the *pattern*. We say that π *avoids* τ , or is τ -*avoiding*, if there is no occurrence of τ in π . The set of all τ -avoiding permutations of all possible sizes including the empty permutation is denoted $\mathfrak{S}(\tau)$. Pattern avoidance proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [5] to singularities of Schubert varieties [6]. A complete study of pattern avoidance for the case $\tau \in \mathfrak{S}_3$ is carried out in [11]. For the case $\tau \in \mathfrak{S}_4$ see [14, 11, 12, 1].

A natural generalization of pattern avoidance is the restricted pattern inclusion, when a prescribed number of occurrences of τ in π is required. Papers [8] and [3] contain simple expressions for the number of permutations containing exactly one 123 and 132 patterns, respectively. The main result of [B2] is that the generating function for the number of permutations containing exactly r 132 patterns is a rational function in variables x and $\sqrt{1-4x}$. This proves a particular case of the general conjecture of Noonan and Zeilberger [9] which is that for any set T of patterns, the sequence of numbers enumerating permutations having a prescribed number of occurrences of patterns in T is P -recursive. Recent paper [10] presents the generating function for the number of 132-avoiding permutations that contain a prescribed number of 123 patterns. The generating function is given in the form of a continued fraction. In the present note we generalize the argument of [10] to get the generating function for the number of 132-avoiding permutations that contain a prescribed number of $12\dots k$ patterns for arbitrary $k \geq 3$. The study of the obtained continued fraction allows us to recover and to generalize the result of [4] that relates the number of 132-avoiding permutations that contain no $12\dots k$ patterns to Chebyshev polynomials of the second kind.

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2. CONTINUED FRACTIONS

Let $f_n^r(k)$ stand for the number of 132-avoiding permutations on n letters that contain exactly r occurrences of $12\dots k$. We denote by $F(x, y; k)$ the generating function of the sequence $\{f_n^r(k)\}$, that is,

$$F(x, y; k) = \sum_{n \geq 0} \sum_{r \geq 0} f_n^r(k) x^n y^r.$$

Our first result is a natural generalization of the main theorem of [10].

Theorem 2.1. *The generating function $F(x, y; k)$ for $k \geq 1$ is given by the continued fraction*

$$F(x, y; k) = \frac{1}{1 - \frac{xy^{d_1}}{1 - \frac{xy^{d_2}}{1 - \frac{xy^{d_3}}{\dots}}}}},$$

where $d_i = \binom{i-1}{k-1}$, and $\binom{a}{b}$ is assumed 0 whenever $a < b$ or $b < 0$.

Proof. Following [10] we define $\eta_j(\pi)$, $j \geq 1$, as the number of occurrences of $12 \dots j$ in π . Define $\eta_0(\pi) = 1$ for any π , which means that the empty pattern occurs exactly once in each permutation. The *weight* of a permutation π is a monomial in k independent variables q_1, \dots, q_k defined by

$$w_k(\pi) = \prod_{j=1}^k q_j^{\eta_j(\pi)}.$$

The *total weight* is a polynomial

$$W_k(q_1, \dots, q_k) = \sum_{\pi \in \mathfrak{S}(132)} w_k(\pi).$$

The following proposition is implied immediately by the definitions.

Proposition 2.1. $F(x, y; k) = W_k(x, 1, \dots, 1, y)$ for $k \geq 2$, and $F(x, y; 1) = W_1(xy)$.

We now find a recurrence relation for the numbers $\eta_j(\pi)$. Let $\pi \in \mathfrak{S}_n$, so that $\pi = (\pi', n, \pi'')$.

Proposition 2.2. *For any $j \geq 1$ and any nonempty $\pi \in \mathfrak{S}(132)$*

$$\eta_j(\pi) = \eta_j(\pi') + \eta_j(\pi'') + \eta_{j-1}(\pi').$$

Proof. Let $l = \pi^{-1}(n)$. Since π avoids 132, each number in π' is greater than any of the numbers in π'' . Therefore, π' is a 132-avoiding permutation of the numbers $\{n-l+1, n-l+2, \dots, n-1\}$, while π'' is a 132-avoiding permutation of the numbers $\{1, 2, \dots, n-l\}$. On the other hand, if π' is an arbitrary 132-avoiding permutation of the numbers $\{n-l+1, n-l+2, \dots, n-1\}$ and π'' is an arbitrary 132-avoiding permutation of the numbers $\{1, 2, \dots, n-l\}$, then $\pi = (\pi', n, \pi'')$ is 132-avoiding. Finally, if (i_1, \dots, i_j) is an occurrence of $12 \dots j$ in π then either $i_j < l$, and so it is also an occurrence of $12 \dots j$ in π'' , or $i_1 > l$, and so it is also an occurrence of $12 \dots j$ in π' , or $i_j = l$, and so (i_1, \dots, i_{j-1}) is an occurrence of $12 \dots j-1$ in π' . The result follows. \square

Now we are able to find the recurrence relation for the total weight W . Indeed, by Proposition 2.2,

$$\begin{aligned}
W_k(q_1, \dots, q_k) &= 1 + \sum_{\emptyset \neq \pi \in \mathfrak{S}(132)} \prod_{j=1}^k q_j^{\eta_j(\pi') + \eta_j(\pi'') + \eta_{j-1}(\pi')} \\
&= 1 + \sum_{\pi' \in \mathfrak{S}(132)} \sum_{\pi'' \in \mathfrak{S}(132)} \prod_{j=1}^k q_j^{\eta_j(\pi'')} \cdot q_1 \prod_{j=1}^{k-1} (q_j q_{j+1})^{\eta_j(\pi')} \cdot q_k^{\eta_k(\pi')} \\
&= 1 + q_1 W_k(q_1, \dots, q_k) W_k(q_1 q_2, q_2 q_3, \dots, q_{k-1} q_k, q_k). \tag{1}
\end{aligned}$$

For any $d \geq 0$ and $1 \leq m \leq k$ define

$$\mathbf{q}^{d,m} = \prod_{j=1}^k q_j^{\binom{d}{j-m}};$$

recall that $\binom{a}{b} = 0$ if $a < b$ or $b < 0$. The following proposition is implied immediately by the well-known properties of binomial coefficients.

Proposition 2.3. *For any $d \geq 0$ and $1 \leq m \leq k$*

$$\mathbf{q}^{d,m} \mathbf{q}^{d,m+1} = \mathbf{q}^{d+1,m}.$$

Observe now that $W_k(q_1, \dots, q_k) = W_k(\mathbf{q}^{0,1}, \dots, \mathbf{q}^{0,k})$ and that by (1) and Proposition 2.3

$$W_k(\mathbf{q}^{d,1}, \dots, \mathbf{q}^{d,k}) = 1 + \mathbf{q}^{d,1} W_k(\mathbf{q}^{d,1}, \dots, \mathbf{q}^{d,k}) W_k(\mathbf{q}^{d+1,1}, \dots, \mathbf{q}^{d+1,k}),$$

therefore

$$W_k(q_1, \dots, q_k) = \frac{1}{1 - \frac{\mathbf{q}^{0,1}}{1 - \frac{\mathbf{q}^{1,1}}{1 - \frac{\mathbf{q}^{2,1}}{\dots}}}}.$$

To obtain the continued fraction representation for $F(x, y; k)$ it is enough to use Proposition 2.1 and to observe that

$$\mathbf{q}^{d,1} \Big|_{q_1=x, q_2=\dots=q_{k-1}=1, q_k=y} = xy \binom{d}{k-1}. \quad \square$$

Remark. For $k = 1$ one recovers from Theorem 2.1 the well-known generating function for the Catalan numbers, $(1 - \sqrt{1 - 4z})/2z$. This result also follows immediately from Proposition 2.1 and equation (1), which for $k = 1$ is reduced to $W_1(q) = 1 + qW_1^2(q)$.

3. CHEBYSHEV POLYNOMIALS

Let us denote by $F_r(x; k)$ the generating function of the sequence $\{f_n^r(k)\}$ for a given r , that is,

$$F_r(x; k) = \sum_{n \geq 0} f_n^r(k) x^n.$$

Recall that $F(x, y; k) = \sum_{r \geq 0} F_r(x; k) y^r$. In this section we find explicit expressions for $F_r(x; k)$ in the case $0 \leq r \leq k$.

Consider a recurrence relation

$$T_j = \frac{1}{1 - xT_{j-1}}, \quad j \geq 1. \quad (2)$$

The solution of (2) with the initial condition $T_0 = 0$ is denoted by $R_j(x)$, and the solution of (2) with the initial condition

$$T_0 = G(x, y; k) = \frac{y}{1 - \frac{xy^{\binom{k}{1}}}{1 - \frac{xy^{\binom{k+1}{2}}}{1 - \frac{xy^{\binom{k+2}{3}}}{\dots}}}}$$

is denoted by $S_j(x, y; k)$, or just S_j when the value of k is clear from the context. Our interest in (2) is stipulated by the following relation, which is an easy consequence of Theorem 2.1:

$$F(x, y; k) = S_k(x, y; k). \quad (3)$$

First of all, we find an explicit formula for the functions $R_j(x)$. Let $U_j(\cos \theta) = \sin(j+1)\theta / \sin \theta$ be the Chebyshev polynomials of the second kind.

Lemma 3.1. *For any $j \geq 1$*

$$R_j(x) = \frac{U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_j\left(\frac{1}{2\sqrt{x}}\right)}. \quad (4)$$

Proof. Indeed, it follows immediately from (2) that $R_j(x)$ is the j th approximant for the continued fraction

$$\frac{1}{1 - \frac{x}{1 - \frac{x}{1 - \frac{x}{\dots}}}}.$$

Hence, by [7, Theorem 2, p. 194], for any $j \geq 1$ one has $R_j(x) = A_j(x)/A_{j+1}(x)$, where

$$A_j(x) = \left(\frac{1 + \sqrt{1-4x}}{2} \right)^j - \left(\frac{1 - \sqrt{1-4x}}{2} \right)^j.$$

Using substitution $x \rightarrow 1/4t^2$ one gets $(2t)^j A_j(1/4t^2) = 2\sqrt{t^2-1}U_{j-1}(t)$, which gives $A_j(x) = \sqrt{1/x-4}x^{j/2}U_{j-1}(1/2\sqrt{x})$, and the result follows. \square

Next, we find an explicit expression for S_j in terms of G and R_j .

Lemma 3.2. *For any $j \geq 1$ and any $k \geq 1$*

$$S_j(x, y; k) = R_j(x) \frac{1 - xR_{j-1}(x)G(x, y; k)}{1 - xR_j(x)G(x, y; k)}. \quad (5)$$

Proof. Indeed, from (2) and $S_0 = G$ we get $S_1 = 1/(1 - xG)$. On the other hand, $R_0 = 0$, $R_1 = 1$, so (5) holds for $j = 1$. Now let $j > 1$, then by induction

$$S_j = \frac{1}{1 - xS_{j-1}} = \frac{1}{1 - xR_{j-1}} \cdot \frac{1 - xR_{j-1}G}{1 - \frac{x(1 - xR_{j-2})R_{j-1}G}{1 - xR_{j-1}}}.$$

Relation (2) for R_j and R_{j-1} yields $(1 - xR_{j-2})R_{j-1} = (1 - xR_{j-1})R_j = 1$, which together with the above formula gives (5). \square

As a corollary from Lemma 3.2 and (3) we get the following expression for the generating function $F(x, y; k)$.

Corollary.

$$F(x, y; k) = R_k(x) + (R_k(x) - R_{k-1}(x)) \sum_{m \geq 1} (xR_k(x)G(x, y; k))^m.$$

Now we are ready to express the generating functions $F_r(x; k)$, $0 \leq r \leq k$, via Chebyshev polynomials.

Theorem 3.1. *For any $k \geq 1$, $F_r(x; k)$ is a rational function given by*

$$F_r(x; k) = \frac{x^{\frac{r-1}{2}} U_{k-1}^{r-1} \left(\frac{1}{2\sqrt{x}} \right)}{U_k^{r+1} \left(\frac{1}{2\sqrt{x}} \right)}, \quad 1 \leq r \leq k,$$

$$F_0(x; k) = \frac{U_{k-1} \left(\frac{1}{2\sqrt{x}} \right)}{\sqrt{x} U_k \left(\frac{1}{2\sqrt{x}} \right)},$$

where U_j is the j th Chebyshev polynomial of the second kind.

Proof. Observe that $G(x, y; k) = y + y^{k+1}P(x, y)$, so from Corollary we get

$$F(x, y; k) = R_k(x) + (R_k(x) - R_{k-1}(x)) \sum_{m=1}^k (xR_k(x))^m y^m + y^{k+1}P'(x, y),$$

where $P(x, y)$ and $P'(x, y)$ are formal power series. To complete the proof, it suffices to use (4) together with the identity $U_{n-1}^2(z) - U_n(z)U_{n-2}(z) = 1$, which follows easily from the trigonometric identity $\sin^2 n\theta - \sin^2 \theta = \sin(n+1)\theta \sin(n-1)\theta$. \square

For the case $r = 0$ this result was proved by a different method in [4].

4. FURTHER RESULTS

There are several ways to generalize the results of the previous sections. First, one can try to get exact formulas for $F_r(x; k)$ in the case $r > k$. The method described in Section 3 allows, in principle, to obtain such formulas, though they become more and more complicated. For example, the following theorem gives an explicit expression for $F_r(x; k)$ when $r \leq k(k+3)/2$.

Theorem 4.1. *For any $k \geq 1$ and $1 \leq r \leq k(k+3)/2$, $F_r(x; k)$ is a rational function given by*

$$F_r(x; k) = \frac{x^{\frac{r-1}{2}} U_{k-1}^{r-1} \left(\frac{1}{2\sqrt{x}} \right)}{U_k^{r+1} \left(\frac{1}{2\sqrt{x}} \right)} \sum_{j=0}^{\lfloor (r-1)/k \rfloor} \binom{r-kj+j-1}{j} \left(\frac{U_k \left(\frac{1}{2\sqrt{x}} \right)}{x^{\frac{k-2}{2k}} U_{k-1} \left(\frac{1}{2\sqrt{x}} \right)} \right)^{kj},$$

where U_j is the j th Chebyshev polynomial of the second kind.

Proof. Indeed, the explicit expression for $G(x, y; k)$ gives

$$G(x, y; k) = y(1 + xy^k + \cdots + x^s y^{ks}) + y^t P(x, y),$$

where $s = \lceil (k+1)/2 \rceil$, $t = 1 + k(k+3)/2$, and $P(x, y)$ is a formal power series. Hence, by Corollary,

$$\begin{aligned} \frac{F(x, y; k) - R_k(x)}{R_k(x) - R_{k-1}(x)} &= \sum_{m \geq 1} (xR_k(x))^m y^m (1 + xy^k + \cdots + x^s y^{ks})^m + y^t P'(x, y) \\ &= \sum_{m \geq 1} (xR_k(x))^m y^m \sum_{j=0}^{ms} \binom{m+j-1}{j} x^j y^{kj} + y^t P'(x, y) \\ &= \sum_{r \geq 1} y^r (xR_k(x))^r \sum_{j=0}^{\lfloor (r-1)/k \rfloor} \frac{\binom{r-kj+j-1}{j} x^j}{(xR_k(x))^{kj}} + y^t P''(x, y), \end{aligned}$$

where $P'(x, y)$ and $P''(x, y)$ are formal power series. The rest of the proof follows the proof of Theorem 3.1. \square

Another possibility is to analyze the case of permutations containing exactly one 132 pattern and r 12... k patterns. Introducing the modified total weight $\Omega_k(q_1, \dots, q_k)$ as the sum of the weights $w_k(\pi)$ over all permutations containing exactly one 132 pattern, we get the following equation:

$$\begin{aligned}\Omega_k(q_1, \dots, q_k) &= q_1 W_k(q_1 q_2, \dots, q_{k-1} q_k, q_k) \Omega_k(q_1, \dots, q_k) \\ &\quad + q_1 W_k(q_1, \dots, q_k) \Omega_k(q_1 q_2, \dots, q_{k-1} q_k, q_k) \\ &\quad + q_1^2 q_2^2 W_k(q_1 q_2, \dots, q_{k-1} q_k, q_k) (W_k(q_1, \dots, q_k) - 1);\end{aligned}$$

for the case $k = 3$ see [10]. By (1) and Proposition 2.3 this is equivalent to

$$\begin{aligned}\Omega_k(\mathbf{q}^{d,1}, \dots, \mathbf{q}^{d,k}) &= \mathbf{q}^{d,1} (\mathbf{q}^{d,2})^2 (W_k(\mathbf{q}^{d,1}, \dots, \mathbf{q}^{d,k}) - 1)^2 \\ &\quad + \mathbf{q}^{d,1} W_k^2(\mathbf{q}^{d,1}, \dots, \mathbf{q}^{d,k}) \Omega_k(\mathbf{q}^{d+1,1}, \dots, \mathbf{q}^{d+1,k}).\end{aligned}\tag{6}$$

Let now $\varphi_n^r(k)$ be the number of permutations on n letters that contain exactly one 132 pattern and r 12... k patterns, and $\Phi_r(x; k)$ be the generating function of the sequence $\{\varphi_n^r(k)\}$ for a given r . In general, equation (6) allows us to find explicit expressions for $\Phi_r(x; k)$. However, they are rather cumbersome, so we restrict ourselves to the case $r = 0$.

Theorem 4.2. *For any $k \geq 3$, $\Phi_0(x; k)$ is a rational function given by*

$$\begin{aligned}\Phi_0(x; k) &= \frac{x}{U_k^2\left(\frac{1}{2\sqrt{x}}\right)} \sum_{j=1}^{k-2} U_j^2\left(\frac{1}{2\sqrt{x}}\right) \\ &= \frac{1}{16 \sin^2(k+1)t \cos^2 t} \left(2k - 5 + 4 \cos^2 t - \frac{\sin(2k-1)t}{\sin t} \right),\end{aligned}$$

where U_j is the j th Chebyshev polynomial of the second kind and $\cos t = 1/2\sqrt{x}$.

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